ON THE STABILITY OF MOTION OF A GYROFRAME

(OB USTOICHIVOSTI DVIZHENIA GIRORAMY)

PMM Vol.25, No.6, 1961, pp. 938-991

D.R. Merkin
(Leningrad)

(Received June 27, 1961)

We consider here the motion of a gyroframe (a rigid body carrying gyroscopes), whose point of suspension is being displaced with respect to an inertial frame of reference in an arbitrarily prescribed manner. Using methods shown in [1], we derive the equations of precessional motion for the whole system. We demonstrate that under certain conditions the equations of motion yield the first integral, which is used to construct the Liapunov function and determine in turn the necessary conditions for the stability of motion. As an example we derive the equations of precessional motion of a horizontal gyrocompass and prove that these equations are equivalent to the equations obtained originally by Ishlinskii [2].

We determine the conditions for the stability of motion of a horizontal gyrocompass (when the velocity of the suspension point v and the angular rotational velocity ω of the trihedron of Darboux are constant).

1. Let us introduce two coordinate systems, the inertial system with the origin at the point O_1 and the system $Ox_1{}^*x_2{}^*x_3{}^*$ in translatory motion with respect to the inertial system. We shall investigate the motion of a system of mass points with respect to these two coordinate systems. Let us denote by $\mathbf{K}$ the principal vector of the angular momentum of our system of mass points about the origin of the inertial reference frame O_1, and by $\mathbf{K}_0'$ the principal vector of the angular momentum of our system about the origin of the translated frame $Ox_1{}^*x_2{}^*x_3{}^*$.

The equation of the angular momentum of our mass points system $d\mathbf{K}/dt = \mathbf{M}^e$ can be easily transformed into

$$\frac{d\mathbf{K}_0'}{dt} = \mathbf{M}_0{}^e + \mathbf{M}_0{}^u \tag{1.1}$$

Here $\mathbf{M}^e$ and $\mathbf{M}_0{}^e$ are the principal moments about the points O_1 and O, respectively, of the external forces acting on the system; $\mathbf{M}_0{}^u$ is the principal moment about the point O of the inertia forces caused by the transport acceleration. All the inertia forces are replaced by a single effective resultant $m\mathbf{w}$ applied at the center of mass of the system (m is the mass of the whole system, $\mathbf{w}$ is the acceleration of the point O).

The kinetic energy of our system can be written in the form

$$T = T' + \tfrac{1}{2}mv^2 + m\mathbf{v}\cdot\mathbf{v}_c' \tag{1.2}$$

In the above equation $\mathbf{v}$ is the velocity of the point O, $\mathbf{v}_c'$ is the velocity of the center of mass of the whole system with respect to the moving axes $Ox_1^*x_2^*x_3^*$; T' is the kinetic energy of the system.

We shall consider now the motion of a gyroframe when the motion of its suspension point O is prescribed. The precessional, or the so-called elementary theory of the motion of a gyroframe can be derived by using two different methods. The first method (see papers by Ishlinskii [2-5]) consists of using the angular-momentum equation in the form (1.1). The precessional theory assumes that at sufficiently high rotational velocities of the gyroscopes the angular momentum $\mathbf{K}_0'$ of the whole system (gyroframe with gyroscopes and other bodies) about the moving axes $Ox_1^*x_2^*x_3^*$ equals the geometric sum of the angular momenta of the gyroscopes $\mathbf{H}_j$

$$\mathbf{K}_0' \approx \mathbf{H} = \sum_{j=1}^{k}\mathbf{H}_j \qquad (k \text{ is the number of gyroscopes})$$

Substituting the approximate value of the angular momentum $\mathbf{K}_0'$ in Equation (1.1), we obtain the equation of precessional motion of the gyroframe

$$\frac{d\mathbf{H}}{dt} = \mathbf{M}_0^{\,e} + \mathbf{M}_0^{\,u} \tag{1.3}$$

To the above vector equation we must add the equations determining the motion of the gyroscopes with respect to the frame.

In the second method of deriving the precessional equations of motion of a gyroframe instead of the angular-momentum equation, we simplify the expression for the kinetic energy. The essence of the second method consists of replacing the kinetic energy of the system with respect to the moving axes $Ox_1^*x_2^*x_3^*$ by the kinetic energy of the gyroscopes rotating about their axes, when the rotational velocities of the gyroscopes are sufficiently high (see, for example, [2]).

2. Let us examine the second method in some detail. Beside the moving axes we introduce in addition two more coordinate systems $Ox_1^\circ x_2^\circ x_3^\circ$ and $Ox_1x_2x_3$. The origins of both systems coincide with the suspension point O, the system $Ox_1x_2x_3$ is fixed in the frame and the system $Ox_1^\circ x_2^\circ x_3^\circ$ rotates with the prescribed angular velocity $\omega_e = \omega_e(t)$ with respect to the trihedron $Ox_1^*x_2^*x_3^*$. Let us denote by ω_{ej}° the components of the angular velocity vector ω_e along the axes x_j; by q_1, q_2, q_3 the Eulerian angles determining the orientation of the gyroframe (the $Ox_1x_2x_3$-coordinate system) with respect to the trihedron $Ox_1^\circ x_2^\circ x_3^\circ$; by q_4, ..., q_n the angles determining the orientation of the gyroscopes' axes with respect to the gyroframe; by a_{js} the cosine of the angle between the vector $\dot{q}_j$ and the axis of the sth gyroscope; by b_{js} the cosine of the angle between the axis x_j° and the axis of the sth gyroscope; by $\dot{\phi}_s$ the rotational angular velocity of a gyroscope; by C_s the moment of inertia of a gyroscope about its axis of symmetry.

When the second method is being used, the kinetic energy (1.2) of the precessional motion of the system consisting of a gyroframe and gyroscopes is replaced by the expression

$$T \approx \frac{1}{2} \sum_{s=1}^{k} C_s \Big(\dot{\varphi}_s + \sum_{j=1}^{n} a_{js}\dot{q}_j + \sum_{j=1}^{3} b_{js}\omega_{ej}^\circ\Big)^2 + \frac{1}{2} m v^2 + m\mathbf{v}\cdot\mathbf{v}_c' \qquad (2.1)$$

In the above equation a_{js} and b_{js} are the known functions of the generalized coordinates q_i, and ω_{ej}° are the prescribed functions of the time. We shall denote by $\boldsymbol{\omega}_r$ the rotational angular velocity of the gyroframe with respect to the trihedron $Ox_1^\circ x_2^\circ x_3^\circ$. The velocity of the center of mass $\mathbf{v}_c'$ with respect to the system $Ox_1^* x_2^* x_3^*$ equals

$$\mathbf{v}_c' = (\boldsymbol{\omega}_r + \boldsymbol{\omega}_e) \times \mathbf{r}_c'$$

Here $\mathbf{r}_c'$ is the radius vector OC of the center of mass. Expression (2.1) can be put in the form

$$T = \frac{1}{2} \sum_{s=1}^{k} C_s \Big(\dot{\varphi}_s + \sum_{j=1}^{n} a_{js}\dot{q}_j + \sum_{j=1}^{3} b_{js}\omega_{ej}^\circ\Big)^2 + T_1 + T_0 \qquad (2.2)$$

where

$$T_1 = m\mathbf{v}\cdot(\boldsymbol{\omega}_r \times \mathbf{r}_c'), \qquad T_0 = m\mathbf{v}\,(\boldsymbol{\omega}_e \times \mathbf{r}_c') + \frac{1}{2} m v^2 \qquad (2.3)$$

Let us mention that T_1 is the linear form of the generalized velocities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$, and T_0 depends only on the generalized coordinates q_i and on the time t.

Taking into consideration that the torques caused by the gyro-motors are equilibrated by the resistance forces, we obtain the k first integrals corresponding to the cyclic coordinates ϕ_s

$$\frac{\partial T}{\partial \dot{\varphi}_s} = C_s\Big(\dot{\varphi}_s + \sum_{j=1}^{n} a_{js}\dot{q}_j + \sum_{j=1}^{3} b_{js}\omega_{ej}^\circ\Big) = H_s \qquad (s = 1, \ldots, k) \qquad (2.4)$$

When constructing the equations of motion of the system it is convenient to eliminate the cyclic coordinates ϕ_s by using the first integrals (2.4). To achieve that we construct the Routh function within an additive constant, in our case

$$R = R_1 + R_0 + T_1 + T_0 \tag{2.5}$$

where

$$R_1 = \sum_{s=1}^{k} H_s \sum_{j=1}^{n} a_{js}\dot{q}_j, \quad R_0 = \sum_{s=1}^{k} H_s \sum_{j=1}^{3} b_{js}\omega_{ej}^{\circ} \tag{2.6}$$

The equations of the precessional motion of the whole system (gyroframe and the gyroscopes) are

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_j} - \frac{\partial R}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, n) \tag{2.7}$$

where Q_j is the generalized force. By (2.5) Equations (2.7) can be put in the form

$$\frac{d}{dt}\frac{\partial R_1}{\partial \dot{q}_j} - \frac{\partial R_1}{\partial q_j} - \frac{\partial R_0}{\partial q_j} = Q_j - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_j} + \frac{\partial T_1}{\partial q_j} + \frac{\partial T_0}{\partial q_j} \tag{2.8}$$

Before going any further, let us make two observations. When constructing the sum $\Sigma a_{js}\dot{q}_j$ in the expression for the kinetic energy (2.1), we consider only those angular velocities $\dot{q}_j$ which are transferred to the sth gyroscope, and when we calculate the generalized forces Q_j we neglect to consider the inertia force caused by the transport acceleration, because the kinetic energy (1.2) or (2.2) refers to the absolute motion.

3. Assuming that the forces acting on the system are derived from a potential, we have

$$Q_j = -\frac{\partial \Pi}{\partial q_j} \qquad (j = 1, \ldots, n) \tag{3.1}$$

where Π is the potential energy. Let also R_0, T_1 and T_2 be explicitly independent of time. With these assumptions Equation (2.8) yields the first integral

$$V = \Pi - T_0 - R_0 = \text{const} \tag{3.2}$$

easily obtained by multiplying each equation in (2.8) by $\dot{q}_j$, summing up the products and taking into consideration that R_1 and T_1 are homogeneous linear functions of the generalized velocities, and that Π, T_0 and R_0 do not depend on the velocities $\dot{q}_i$.

We shall assume that the displacements of the coordinate system $Ox_1^{\circ}x_2^{\circ}x_3^{\circ}$ and that the forces acting on the system are such that the motion described by the equations $q_j = 0$ $(j = 1, 2, \ldots, n)$ is possible, For such a motion Equations (2.8) become applicable for the perturbations, and if the function

$$W = V - V(0) \tag{3.3}$$

is sign-definite, then the unperturbed precessional motion $q_j = 0$ is stable by the Liapunov criterion.

4. Let us consider as an example the derivation of the equations of motion for a horizontal gyrocompass and of the conditions of its stability. The basic scheme of the gyrocompass system can be found in [2], where the author derives the equations of motion using the equations of moments. In our derivations we are using essentially the same method as in [2].

The earth is assumed to be a perfect sphere and the suspension point O of the gyroframe moves on the earth's surface. Let us introduce a coordinate system $Ox^*y^*z^*$ which is in translatory motion and whose axes are oriented on distant stars, and a geographically oriented system $O\xi\eta\zeta$. Let U be the earth's angular velocity, ϕ be the geographical latitude of the gyrocompass, V_E and V_N be respectively the known eastern and northern components of the velocity of the suspension point with respect to the earth's surface. The angular velocity of the latitude variation $\dot{\phi}$ and of the longitude variation $\dot{\lambda}$ equal

$$\dot{\varphi} = \frac{V_N}{R}, \qquad \dot{\lambda} = \frac{V_E}{R\cos\varphi} \tag{4.1}$$

where R is the earth's radius. The angular velocity vector of the latitude variation is directed west, and the angular velocity vector of the longitude variation is parallel to the earth's axis.

The angular velocity vector of the rotation of $O\xi\eta\zeta$ with respect to the star-oriented system is the resultant of the earth's rotation angular velocity vector and of the angular velocity vectors of the latitude variation and longitude variation. Its ξ, η and ζ components equal

$$u_\xi = -\dot{\varphi}, \qquad u_\eta = (U + \dot{\lambda})\cos\varphi, \qquad u_\zeta = (U + \dot{\lambda})\sin\varphi \tag{4.2}$$

The velocity v of the suspension point O with respect to the inertial reference frame whose origin is at the earth's center (the acceleration of the earth's center is neglected) is the resultant of the transport velocity $RU\cos\phi$, directed east along the tangent to a meridian, and of the relative velocity whose components are V_E, V_N, O. Consequently, the velocity v equals (Fig. 1)

$$v = \sqrt{V_N{}^2 + (V_E + RU\cos\varphi)^2} \tag{4.3}$$

Following [2], we introduce a new coordinate system $Ox^\circ y^\circ z^\circ$ whose z-axis (Fig. 1) coincides with the ζ-axis, and the x°-axis is along the velocity vector v. The y°-axis must then be directed along the resultant of u_ξ and u_η. The angle ϑ between the axes ξ and x° is given by the formula

$$\operatorname{tg}\vartheta = \frac{\dot{\varphi}}{(U + \dot{\lambda})\cos\varphi} = \frac{V_N}{V_E + RU\cos\varphi} \tag{4.4}$$

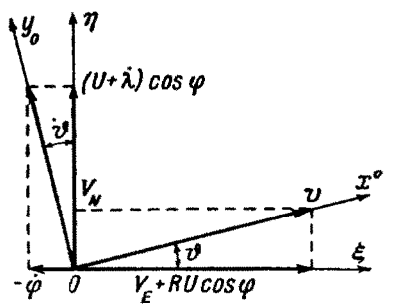

Fig. 1.

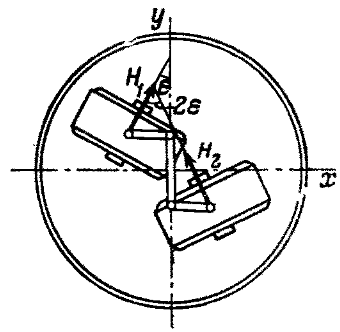

Fig. 2.

Using Fig. 1 and Formulas (4.1) to (4.4), we calculate the x°-, y°-, and z°-components of the rotational angular velocity ω_e of the system $x^\circ y^\circ z^\circ$ with respect to $x^* y^* z^*$

$$\omega_{ex^\circ} = 0, \qquad \omega_{ey^\circ} = \frac{v}{R}, \qquad \omega_{ez^\circ} = \omega \tag{4.5}$$

where

$$\omega = (U + \dot{\lambda})\sin\varphi + \dot{\vartheta} = \frac{V_E}{R}\tan\varphi + U\sin\varphi + \dot{\vartheta} \tag{4.6}$$

We introduce finally the coordinate system xyz fixed in the gyroframe, whose z-axis is parallel to the rotation axes of the gyroscopes' casings (inner gimbal rings), the y-axis makes equal angles with both gyroscopes' axes, consequently the direction of the x-axis is uniquely determined (Fig. 2). (Figure 2 is given without explanations, which can be found for example in [2].) The orientation of the gyroframe with respect to $Ox^\circ y^\circ z^\circ$ is determined by the three angles α, β and γ (Fig. 3). The direction cosines of the angles between $Oxyz$ and $Ox^\circ y^\circ z^\circ$ are listed in the following table:

	x°	y°	z°
x	$\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma$	$\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$	$-\cos\beta\sin\gamma$
y	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$	$\sin\beta$
z	$\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma$	$\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$	$\cos\beta\cos\gamma$

(4.7)

The x-, y-, and z-components of the angular velocity vector ω_e and of the angular velocity vector ω_r of the gyroframe in rotation with respect to $Ox^\circ y^\circ z^\circ$ are

$$\begin{aligned}\omega_{ex} &= \frac{v}{R}(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) - \omega\cos\beta\sin\gamma\\ \omega_{ey} &= \frac{v}{R}\cos\alpha\cos\beta + \omega\sin\beta \\ \omega_{ez} &= \frac{v}{R}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + \omega\cos\beta\cos\gamma\end{aligned} \tag{4.8}$$

$$\omega_{rx} = -\dot\alpha\cos\beta\sin\gamma + \dot\beta\cos\gamma, \qquad \omega_{ry} = \dot\alpha\sin\beta + \dot\gamma \tag{4.9}$$
$$\omega_{rz} = \dot\alpha\cos\beta\cos\gamma + \dot\beta\sin\gamma$$

We need also the y-component of the angular velocity vector of the gyroframe's rotation with respect to the inertial system (the remaining components are not needed).

We have the relation $\omega_y = \omega_{ry} + \omega_{ey}$, and by (4.8) and (4.9)

$$\omega_y = \dot\alpha\sin\beta + \dot\gamma + \frac{v}{R}\cos\alpha\cos\beta + \omega\sin\beta \tag{4.10}$$

In order to obtain in the Routh function the expression for $R_1 + R_0$ (see, for example, [1] and also Formula (2.6)) we must: (1) calculate the sums of the projections of angular velocities on the axes of each gyroscope (excluding the velocity of spin); (2) multiply these sums by the angular momenta of the corresponding gyroscopes and add together these products.

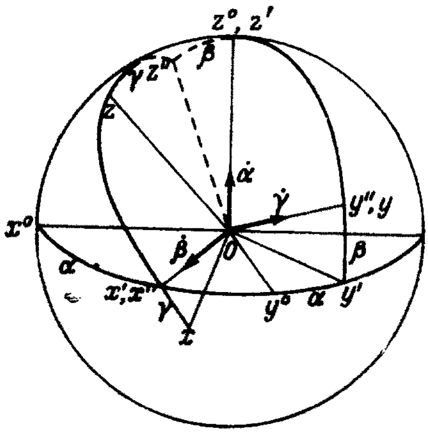

Fig. 3.

Applying the above procedure to the system which is shown in Fig. 2, we obtain

$$R_1 + R_0 = H_1(\omega_x\sin\varepsilon + \omega_y\cos\varepsilon) + H_2(-\omega_x\sin\varepsilon + \omega_y\cos\varepsilon)$$

Assuming that angular momenta of the two gyroscopes equal each other ($H_1 = H_2 = H$; an angular momentum in [2] is denoted by B), and by (4.10), we obtain

$$R_1 + R_0 = 2H\cos\varepsilon\left(\dot\alpha\sin\beta + \dot\gamma + \frac{v}{R}\cos\alpha\cos\beta + \omega\sin\beta\right)$$

Hence

$$R_1 = 2H\cos\varepsilon\,(\dot\alpha\sin\beta + \dot\gamma), \quad R_0 = 2H\cos\varepsilon\left(\frac{v}{R}\cos\alpha\cos\beta + \omega\sin\beta\right) \tag{4.11}$$

We are going to calculate now T_1 and T_0 by Formulas (2.3). The x°-, y°- and z°-components of the velocity vector $\mathbf{v}$ of the suspension point equal

$$v_{x^\circ} = v, \qquad v_{y^\circ} = 0, \qquad v_{z^\circ} = 0$$

The x-, y-, and z-components of the vector $\mathbf{v}$ are found by Table (4.7) and equal

$$v_x = v(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma), \qquad v_y = -v\sin\alpha\cos\beta$$

$$v_z = v(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) \tag{4.12}$$

The center of gravity of the whole system is on the x-axis at the distance l below the suspension point. Therefore, the x-, y-, and z-components of the vector $\mathbf{r}_c'$ are

$$x_c = 0, \qquad y_c = 0, \qquad z_c = -l \tag{4.13}$$

Performing multiplications as shown previously in (2.3), and by (4.8), (4.9), (4.12), and (4.13), we obtain

$$T_1 = mlv[(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma)\dot\alpha - \sin\alpha\cos\beta\cos\gamma\dot\beta -$$
$$- (\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma)\dot\gamma] \tag{4.14}$$
$$T_0 = \frac{1}{2}mv^2 + ml\omega v(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) - ml\frac{v^2}{R}\cos\beta\cos\gamma$$

We shall calculate now the generalized forces. The gyroframe is under the action of the gravity force F directed toward the earth's center along the earth's radius. The potential energy due to F equals $\Pi_1 = F\zeta$, where ζ is the coordinate of the center of gravity of the gyroframe with all the bodies attached to it. By Equation (4.13) and by Table (4.7) we obtain

$$\Pi_1 = -Fl\cos\beta\cos\gamma \tag{4.15}$$

A gyroframe has an internal device which generates the moment $N(\epsilon)$ about the rotation axes of the gyroscopes' casings. If we neglect all other forces except the ones under consideration, we obtain the following generalized forces for our system:

$$Q_\alpha = -\frac{\partial\Pi_1}{\partial\alpha} = 0, \qquad Q_\beta = -\frac{\partial\Pi_1}{\partial\beta} = -Fl\sin\beta\cos\gamma \tag{4.16}$$
$$Q_\gamma = -\frac{\partial\Pi_1}{\partial\gamma} = -Fl\cos\beta\sin\gamma, \qquad Q_\varepsilon = -N(\varepsilon)$$

Using Expressions (2.8) for the angles α, β, γ, and ϵ, and by (4.11), (4.14), and (4.16), we obtain the equations of motion for our system (the right members of the first three equations can be obtained as the sum of the moments of the gravity force F and of the resultant of inertia forces mv about the z_0-, x'- and y-axes (Fig. 3)):

$$
\begin{gathered}
\frac{d}{dt}(2H\cos\varepsilon)\sin\beta + 2H\cos\varepsilon\cdot\cos\beta\dot\beta + 2H\cos\varepsilon\frac{v}{R}\sin\alpha\cos\beta = \\
= -ml\frac{dv}{dt}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + ml\omega v(\cos\alpha\sin\gamma + \\
+\sin\alpha\sin\beta\cos\gamma) \\
-2H\cos\varepsilon\left(\dot\alpha\cos\beta - \frac{v}{R}\cos\alpha\sin\beta + \omega\cos\beta\right) = ml\frac{dv}{dt}\sin\alpha\cos\beta\cos\gamma - \\
-ml\omega v\cos\alpha\cos\beta\cos\gamma - \left(F - m\frac{v^2}{R}\right)l\sin\beta\cos\gamma \\
\frac{d}{dt}2H\cos\varepsilon = ml\frac{dv}{dt}(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) + \\
+ml\omega v(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) - \left(F - m\frac{v^2}{R}\right)l\cos\beta\sin\gamma \\
2H\sin\varepsilon\left(\dot\alpha\sin\beta + \dot\gamma + \frac{v}{R}\cos\alpha\cos\beta + \omega\sin\beta\right) = -N(\varepsilon)
\end{gathered}
\tag{4.17}
$$

Equations (4.17) are equivalent to the equations (40) in [2] which were obtained by a different procedure. Indeed, the second and the fourth equations of the system (40) in [2] coincide with the third and the fourth equations of (4.17). In order to obtain the first equation of (40), we reduce the first equation of (4.17) by the third equation of (4.17) to the form

$$
\begin{gathered}
2H\cos\varepsilon\left(\dot\beta + \frac{v}{R}\sin\alpha\right) = -ml\frac{dv}{dt}\sin\alpha\cos\beta\sin\gamma + \\
+ml\omega v\cos\alpha\cos\beta\sin\gamma + \left(F - m\frac{v^2}{R}\right)l\sin\beta\sin\gamma
\end{gathered}
$$

then we multiply this equation by $\sin\gamma$ and multiply the second equation (4.17) by $\cos\gamma$ and add the two products. The third equation of (40) can be easily obtained by some other linear combinations of equations in (4.17).

5. Our gyroframe becomes a horizontal gyrocompass if we select the moment $N(\epsilon)$ permitting the motion determined by $\alpha = \beta = \gamma = 0$, that is such a motion when the equatorial plane of the gyroframe $Oxyz$ remains all the time horizontal (gyrohorizon), and the y-axis points north within a course correction angle ϑ (gyrocompass). Substituting the above values for the angles α, β and γ in Equations (4.17), we obtain

$$
2H\cos\varepsilon\cdot\omega = ml\omega v, \qquad 2H\sin\varepsilon\frac{v}{R} = -N(\varepsilon)
$$

(см. [2])

$$
\begin{gathered}
2H\cos\varepsilon = mlv = ml\sqrt{V_N^2 + (V_E + RU\cos\varphi)^2} \\
N(\varepsilon) = -\frac{4H^2}{mlR}\cos\varepsilon\sin\varepsilon
\end{gathered}
\tag{5.1}
$$

If the above conditions are satisfied and if at the initial instant of time $\alpha = \beta = \gamma = 0$, then the gyroframe will be in equilibrium relative to $Ox^\circ y^\circ z^\circ$ showing all the time both north and the local vertical.

6. We shall investigate now the stability of motion, assuming that the functions R_0, T_0 and T_1 do not depend explicitly on the time (this is equivalent to the assumption that v and ω are constants). Let ϵ_0 be the value of the angle ϵ at which the first condition (5.1) is exactly satisfied:

$$2H \cos \varepsilon_0 = mlv \tag{6.1}$$

Let us introduce through the equation

$$\varepsilon = \varepsilon_0 + \delta$$

the new angle δ. The generalized force $-N(\epsilon)$ will be conservative and its potential energy will be

$$\Pi_2 = \int N(\varepsilon)\, d\varepsilon = \frac{H^2}{mlR} \cos 2\varepsilon \tag{6.2}$$

Using the potential energy of the force F (4.15), we find the potential energy Π of the whole system

$$\Pi = -Fl \cos\beta \cos\gamma + \frac{H^2}{mlR} \cos 2\varepsilon \tag{6.3}$$

Under these assumptions the first integral (3.2) exists and in our case (see (4.11), (4.14) and (6.3)) it has the form

$$V = -l\left(F - m\frac{v^2}{R}\right)\cos\beta \cos\gamma + \frac{H^2}{mlR}\cos 2(\varepsilon_0 + \delta) - \frac{1}{2}mv^2 -$$
$$- ml\omega v - (\sin\alpha \sin\gamma - \cos\alpha \sin\beta \cos\gamma) - 2H\cos(\varepsilon_0 + \delta) \times$$
$$\times \left(\frac{v}{R}\cos\alpha \cos\beta + \omega \sin\beta\right) \tag{6.4}$$

We consider the difference $V - V(0)$ and expand it in power series of α, β, γ and δ. By Equation (6.1) and after certain transformations we find

$$V - V(0) = \frac{l}{2}\left[\frac{mv^2}{R}\alpha^2 + F\beta^2 + \left(F - \frac{mv^2}{R}\right)\gamma^2 + m\frac{v^2}{R}\tan^2\varepsilon_0\delta^2 - \right.$$
$$\left. - 2m\omega v\beta\gamma + 2m\omega v \tan \varepsilon_0\beta\delta\right] + \ldots \tag{6.5}$$

The dots which follow the last term indicate higher-order terms which were neglected.

Let us take the quadratic form inside the brackets and apply to it the criterion of Sylvester. We find that if the condition

$$F - m\frac{v^2}{R} - mR\omega^2 > 0 \tag{6.6}$$

is satisfied, then for sufficiently small values of α, β, γ and δ the function $W = V - V(0)$ is positive-definite. Its derivative, on the strength of the perturbed equations, equals zero ($W =$ constant), hence, by Liapunov's theorem, the unperturbed motion of the horizontal gyrocompass $\alpha = \beta = \gamma = \delta = 0$ is stable.

Let us examine quickly the condition (6.6). If we assume, following [2], that

$$F - m\frac{v^{2'}}{R} = mg$$

is approximately satisfied, then the inequality will take the form

$$\omega < \nu \left(\nu = \sqrt{\frac{g}{R}}\right) \qquad (6.7)$$

where ν is the frequency corresponding to a period of Schuler [6].

7. We shall now take into account the resistance forces. If we do that we have instead of the integral (6.5)

$$\frac{d}{dt}(V - V_0) = -(a\dot{\alpha}^2 + b\dot{\beta}^2 + c\dot{\gamma}^2 + d\dot{\delta}^2$$

where a, b, c and d are arbitrarily small constants which characterize the dissipative forces.

When $\omega < \nu$, then the unperturbed motion $\alpha = \beta = \gamma = \delta = 0$ is asymptotically stable, and when $\omega > \nu$ then the function $V - V_0$ can take on negative values, and the motion is unstable. If we take into account resistance forces in every part of the system, then, with all other assumptions, the inequality $\omega < \nu$ is not only sufficient but also the necessary condition of stability of motion of a horizontal gyrocompass.

BIBLIOGRAPHY

1. Merkin, D.R., *Giroskopicheskie sistemy* (*Gyroscopic Systems*). GITTL, 1956.

2. Ishlinskii, A.Iu., K teorii girogorizont kompasa (On the theory of a horizontal gyrocompass). *PMM* Vol. 20, No. 4, 1956.

3. Ishlinskii, A.Iu., K teorii giroskopicheskogo maiatnika (On the theory of a gyroscopic pendulum). *PMM* Vol. 21, No. 1, 1957.

4. Ishlinskii, A.Iu., Teoriia dvugiroskopicheskoi vertikali (Theory of a two-gyroscopes vertical). *PMM* Vol. 21, No. 2, 1957.

5. Ishlinskii, A.Iu., K teorii slozhnykh sistem giroskopicheskoi stabilizatsii (On the theory of complicated systems of gyroscopic stability). *PMM* Vol. 22, No. 3, 1958.

6. Zhbanov, Iu.K., Issledovanie svobodnykh kolebanii v sisteme avtonomnogo opredeleniia koordinat dvizhushchegosia ob'ekta (Investigations of free oscillations in autonomous system of coordinates for a moving object). *PMM* Vol. 24, No. 6, 1960.

Translated by T.L.